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REGULAR INTEGRAL EQUATIONS FOR THE SECOND BOUNDARY-VALUE PROBLEM OF THE BENDING OF AN ANISOTROPIC ELASTIC PLATE

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Two systems of Fredholm equations of the second kind are constructed for the solution of the second boundary-value problem of the bending of an anisotropic plate (a normal bending moment and a generalized shear force are specified on the boundary of the simply-connected domain) under the assumption of validity of the Kirchhoff–Love hypotheses. Correct equilibrium conditions are specified for the examined boundary-value problem.

Key words: *anisotropy, plate, Fredholm equations of the second kind.*

In this paper, regular integral equations are constructed for the solution of the second boundary-value problem of the bending of an anisotropic plate in a simply-connected bounded domain with Lyapunov's boundary. A normal bending moment and a generalized shear force are specified on the boundary of the domain. This boundary-value problem does not belong to the class of uniquely solvable problems since combinations of the second and third derivatives of the solution are specified on the boundary and the deflection equation is of the fourth order. Therefore, the deflection is determined only with accuracy up to a linear function of the coordinates and in this sense, we have analogy with the first boundary-value problem in elasticity theory (on the boundary, the force vector is specified). According to the theory of partial differential equations, this problem is coupled to the first boundary-value problem (on the boundary, the deflection and its normal derivative are specified). For the first boundary-value problem, a system of regular integral equations was constructed previously [1] with minimum constraints on the boundary of the domain and the boundary data in the Hölder class of functions. It was of interest to construct a similar system of equations for the coupled problem. The existence of a unique (with accuracy up to a linear function) generalized solution for the coupled problem was proved in [2] for an isotropic material. The existence of a generalized solution for an anisotropic material directly follows from the methods described in [2]. The construction of a system of regular equations for the examined boundary-value problem is complicated by the fact that the expression of the generalized shear force includes the derivative with respect to the arc length, and, strictly speaking, the functions specifying the boundary of the domain should have two continuous derivatives. Lekhnitskii [3] proposed an original approach to overcome this difficulty in the theory of anisotropic plates. Using this approach and another method, which has been applied earlier in elasticity theory [4], it was possible to construct a system of regular integral equations for the examined boundary-value problems in a limited domain with Lyapunov's boundary for an anisotropic material. In this case, the limiting transition to an isotropic material is possible.

1. The homogeneous equation of plate bending in divergent form is written as

$$L(w(x_1,x_2)) = \frac{\partial^2}{\partial x_1^2} M_{11} + 2\,\frac{\partial^2}{\partial x_1\,\partial x_2} M_{12} + \frac{\partial^2}{\partial x_2^2} M_{22} = 0. \tag{1.1}$$

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The bending moments M_{ij} $(i, j = 1, 2)$ are calculated from the formulas

$$-M_{11} = D_{11}\,\frac{\partial^2 w}{\partial x_1^2} + D_{12}\,\frac{\partial^2 w}{\partial x_2^2} + 2D_{16}\,\frac{\partial^2 w}{\partial x_1\,\partial x_2},$$

$$-M_{22} = D_{12}\,\frac{\partial^2 w}{\partial x_1^2} + D_{22}\,\frac{\partial^2 w}{\partial x_2^2} + 2D_{26}\,\frac{\partial^2 w}{\partial x_1\,\partial x_2},$$

$$-M_{12} = D_{16}\,\frac{\partial^2 w}{\partial x_1^2} + D_{26}\,\frac{\partial^2 w}{\partial x_2^2} + 2D_{66}\,\frac{\partial^2 w}{\partial x_1\,\partial x_2}.$$

Here D_{ij} $(i, j = 1, 2, 6)$ is the flexural rigidity and $w(x_1, x_2)$ is the plate deflection. The shear forces are given by

$$N_{11} = \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2}, \qquad N_{22} = \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2}.$$

Of course, the flexural-rigidity matrix is assumed to positively defined. We write Green's formula for the examined boundary-value problem. Multiplying (1.1) by the function $v(x_1, x_2)$ and performing double integration by parts, we obtain

$$\int\limits_Q L(w)v(x_1,x_2)\,dx_1\,dx_2 = \int\limits_Q \sum_{i,j=1}^{2} M_{ij}(w)\,\frac{\partial^2 v}{\partial x_i\,\partial x_j}\,dx_1\,dx_2$$

$$+ \int\limits_{\partial Q} (N_{11}(w)n_1 + N_{22}(w)n_2)v\,ds - \int\limits_{\partial Q} \sum_{i,j=1}^{2} M_{ij}(w)\,\frac{\partial v}{\partial x_j}\,n_i\,ds. \tag{1.2}$$

Here $\boldsymbol{n} = (n_1, n_2) = (-x_2'(s), x_1'(s))$ is the inward normal vector to the boundary ∂Q of the simply-connected domain Q. The domain is oriented so that its interior remains on the left in counterclockwise circulation around the boundary. It is assumed that the coordinate origin is inside the domain. The derivatives with respect to the normal and tangent to the boundary can be written as

$$\frac{\partial v}{\partial n} = \frac{\partial v}{\partial x_1}\,n_1 + \frac{\partial v}{\partial x_2}\,n_2, \qquad \frac{\partial v}{\partial s} = \frac{\partial v}{\partial x_1}\,n_2 - \frac{\partial v}{\partial x_2}\,n_1. \tag{1.3}$$

Let us substitute (1.3) into (1.2). Then,

$$-\int\limits_{\partial Q} \sum_{i,j=1}^{2} M_{ij}(w)\,\frac{\partial v}{\partial x_j}\,n_i\,ds = -\int\limits_{\partial Q} \Big(M_n(w)\,\frac{\partial v}{\partial n} + M_t(w)\,\frac{\partial v}{\partial s}\Big)\,ds.$$

Here

$$M_n(w) = M_{11}n_1^2 + 2M_{12}n_1n_2 + M_{22}n_2^2; \qquad M_t(w) = (M_{11} - M_{22})n_1n_2 + M_{12}(n_2^2 - n_1^2).$$

Let us integrate the term that contains the tangential derivative with respect to the function $v(x_1, x_2)$ once again by parts along the boundary taking into account that the second derivatives of the function $w(x_1, x_2)$ are single-valued. Then, Green's formula becomes

$$\int\limits_Q L(w)v(x_1,x_2)\,dx_1\,dx_2 = \int\limits_Q \sum_{i,j=1}^{2} M_{ij}(w)\,\frac{\partial^2 v}{\partial x_i\,\partial x_j}\,dx_1\,dx_2$$

$$+ \int\limits_{\partial Q} \Big[(N_{11}(w)n_1 + N_{22}(w)n_2) + \frac{\partial M_t(w)}{\partial s}\Big]v\,ds - \int\limits_{\partial Q} M_n(w)\,\frac{\partial v}{\partial n}\,ds. \tag{1.4}$$

For Eq. (1.1), we formulate the boundary-value problem

$$N(w) = N_{11}(w)n_1 + N_{22}(w)n_2 + \frac{\partial M_t(w)}{\partial s}\Big|_{\partial Q} = h_1(s); \tag{1.5}$$

$$-M_n(w)\Big|_{\partial Q} = h_2(s). \tag{1.6}$$

It is assumed that $h_1(s) \in C^{0,\lambda}(\partial Q)$ and $h_2(s) \in C^{1,\lambda}(\partial Q)$. We recall that by definition, $g(s) \in C^{k,\lambda}(\partial Q)$ if $g(s)$ has k continuous derivatives and the derivative of order k satisfies the Hölder condition with an exponent λ ($0 < \lambda < 1$). Let us now find the necessary conditions on the boundary data for the existence of a solution of the boundary-value problem (1.1), (1.5), (1.6). Let $w(x_1, x_2)$ be a solution of problem (1.1), (1.5), (1.6), and $v(x_1, x_2)$ be a solution of the homogeneous problem. It is obvious that the second derivatives of the function v are equal to zero and, hence, $v(x_1, x_2) = ax_1 + bx_2 + c$. For the substitution of

v into (1.4), the arbitrariness of the constants a, b, and c implies that the boundary data should obey the following relations (equilibrium conditions):

$$\int\limits_{\partial Q} h_1(s)\, ds = 0; \tag{1.7}$$

$$\int\limits_{\partial Q} (h_1(s)x_1(s) + h_2(s)x_2'(s))\, ds = 0; \tag{1.8}$$

$$\int\limits_{\partial Q} (h_1(s)x_2(s) - h_2(s)x_1'(s))\, ds = 0. \tag{1.9}$$

These conditions are necessary and sufficient for the existence of a unique (with accuracy up to a linear function) solution of the boundary-value problem (1.1), (1.5), (1.6). Conditions (1.7)–(1.9) can be written differently. We set

$$g_1(s) = \int\limits_0^s h_1(t)\, dt, \qquad g_2(s) = h_2(s).$$

Then, condition (1.7) is obviously equivalent to the relation $g_1(0) = g_1(L) = 0$, and conditions (1.8) and (1.9) become

$$\int\limits_{\partial Q} (-g_1(s)x_1'(s) + g_2(s)x_2'(s))\, ds = 0, \qquad \int\limits_{\partial Q} (g_1(s)x_2'(s) + g_2(s)x_1'(s))\, ds = 0 \tag{1.10}$$

(L is the length of the boundary). Conditions (1.10) can also be written in equivalent form (1.10):

$$\int\limits_{\partial Q} (g_1(s)n_2 + g_2(s)n_1)\, ds = 0, \qquad \int\limits_{\partial Q} (-g_1(s)n_1 + g_2(s)n_2)\, ds = 0.$$

2. We recall that the solution of the homogeneous equation (1.1) can be represented as the sum of two analytical functions:

$$w(x_1, x_2) = \mathrm{Re}\,(\varphi_1(z_1) + \varphi_2(z_2)).$$

Here $\varphi_k(z_k) = \varphi_k(x_1 + \mu_k x_2)$ ($k = 1, 2$) are analytical functions of the arguments and μ_k ($\mathrm{Im}\,\mu_k > 0$, $k = 1, 2$) are the complex parameters of the material determined from the characteristic equation

$$g(\mu) = D_{22}\mu^4 + 4D_{26}\mu^3 + 2(D_{12} + 2D_{66})\mu^2 + 4D_{16}\mu + D_{11} = 0.$$

The quantities M_{ij} ($i, j = 1, 2$) have the form

$$-M_{11} = \mathrm{Re}\sum_{k=1}^{2} p_k\varphi_k''(z_k), \qquad -M_{12} = \mathrm{Re}\sum_{k=1}^{2} r_k\varphi_k''(z_k), \qquad -M_{22} = \mathrm{Re}\sum_{k=1}^{2} q_k\varphi_k''(z_k),$$

where $p_k = D_{11} + D_{12}\mu_k^2 + 2D_{16}\mu_k$, $q_k = D_{12} + D_{22}\mu_k^2 + 2D_{26}\mu_k$, $r_k = D_{16} + D_{26}\mu_k^2 + 2D_{66}\mu_k$ ($k = 1, 2$). Then,

$$-M_n = \mathrm{Re}\sum_{k=1}^{2}(p_k n_1^2 + 2n_1n_2r_k + q_kn_2^2)\varphi_k''(z_k).$$

Let $l_k = p_kn_1^2 + q_kn_2^2 + 2r_kn_1n_2$, where $k = 1, 2$. It is easy to show that the functions l_k ($k = 1, 2$) are proportional to $t_k(s) = x'(s) + \mu_k y'(s) = n_2 - \mu_k n_1$ ($k = 1, 2$). Indeed, dividing l_k by $n_2 - \mu_k n_1$, we obtain the equality

$$l_k = (n_2 - \mu_k n_1)(n_2 q_k + n_1(2r_k + q_k\mu_k)) + n_1^2(p_k + 2r_k\mu_k + q_k\mu_k^2).$$

The multiplier at n_1^2 is equal to zero since

$$g(\mu_k) = p_k + 2\mu_k r_k + \mu_k^2 q_k = 0, \qquad k = 1, 2.$$

Let

$$m_k = n_2 q_k - n_1 p_k/\mu_k, \qquad k = 1, 2.$$

Then $-M_n$ is written as

$$-M_n = \mathrm{Re}\Big(m_1(n_2 - \mu_1 n_1)\varphi_1''(z_1) + m_2(n_2 - \mu_2 n_1)\varphi_2''(z_2)\Big). \tag{2.1}$$

Let us revert to boundary condition (1.6). We introduce the function

$$G(x_1, x_2) = N_{11}n_1 + N_{22}n_2.$$

It is easy to see that

$$G(x_1, x_2) = \mathrm{Re}\sum_{k=1}^{2}(r_k + \mu_k q_k)(n_2 - \mu_k n_1)\varphi_k'''(z_k). \tag{2.2}$$

Hence, $G(x_1, x_2)$ can be written as

$$G(x_1, x_2) = \frac{\partial}{\partial s}\ \mathrm{Re}\sum_{k=1}^{2}(r_k + \mu_k q_k)\varphi_k''(z_k) = \frac{\partial M}{\partial s},$$

where

$$M(x_1, x_2) = \mathrm{Re}\sum_{k=1}^{2}(r_k + \mu_k q_k)\varphi_k''(z_k).$$

Therefore,

$$G + \frac{\partial M_t}{\partial s} = \mathrm{Re}\sum_{k=1}^{2}(r_k + \mu_k q_k + (p_k - q_k)n_1 n_2 + r_k(n_2^2 - n_1^2))\varphi_k'''(z_k),$$

and

$$r_k + \mu_k q_k + (p_k - q_k)n_1 n_2 + r_k(n_2^2 - n_1^2) = -(n_2 - \mu_k n_1)(q_k n_1 + p_k n_2/\mu_k).$$

As a result, boundary condition (1.5) can be written as

$$\frac{\partial\,(M + M_t)}{\partial s} = h_1(s).$$

This relation can be examined as an ordinary differential equation for the sum $M + M_t$. Because the specified function $h_1(s)$ is considered a periodic function of the arc length, it follows that for the unique determination of the sum $M + M_t$, the function $h_1(s)$ should satisfy the condition

$$\int_{\partial Q} h_1(s)\,ds = 0,$$

i.e., condition (1.7). This condition is also a sufficient one. Consequently, boundary conditions (1.5) and (1.6) can be written as

$$-(M + M_t) = \mathrm{Re}\sum_{k=1}^{2}(n_2 - \mu_k n_1)\Big(n_1 q_k + n_2\,\frac{p_k}{\mu_k}\Big)\varphi_k''(z_k)\Big|_{\partial Q} = \int_0^s h_1(t)\,dt + C_1; \tag{2.3}$$

$$M_n = \mathrm{Re}\sum_{k=1}^{2}(n_2 - \mu_k n_1)\Big(n_2 q_k - n_1\,\frac{p_k}{\mu_k}\Big)\varphi_k''(z_k)\Big|_{\partial Q} = h_2(s)\,ds \tag{2.4}$$

(C_1 is a real constant). We shall seek $\varphi_k''(z_k)$ $(k = 1, 2)$ in the form of Cauchy-type integrals:

$$\varphi_k''(z_k) = \frac{1}{\pi i}\int_{\partial Q}\frac{\alpha_k\,ds}{t_k - z_k}, \qquad k = 1, 2.$$

It is obvious that

$$\varphi_k''(z_k) = \frac{1}{\pi i}\int\limits_{\partial Q} \frac{\alpha_k (t_k'(s))^{-1}\, ds}{t_k - z_k}, \qquad k = 1, 2.$$

Here α_k $(k = 1, 2)$ are the unknown densities. We assume that they satisfy the Hölder condition, i.e., there exists a constant λ $(0 < \lambda < 1)$ such that for any $s, s_0 \in \partial Q$, the following inequality is valid:

$$|\alpha_k(s) - \alpha_k(s_0)| < C|s - s_0|^{\lambda}, \qquad k = 1, 2.$$

In this assumption for the Cauchy type integral, the Sokhotsky formula holds, and, hence, when the point $z = x_1 + ix_2$ tends to the point $t(s) = x_1(s) + ix_2(s)$ from inside the domain $Q = Q_i$, we obtain

$$\lim_{z \to t(s_0)} \varphi_k''(z_k) = \alpha_k(s_0)(t_k'(s_0))^{-1} + \frac{1}{\pi i}\int\limits_{\partial Q} \frac{\alpha_k (t_k'(s))^{-1}\, ds}{t_k - t_{k0}}, \qquad k = 1, 2. \tag{2.5}$$

Here $t_k = x_1(s) + \mu_k x_2(s)$ and $t_{k0} = x_1(s_0) + \mu_k x_2(s_0)$ $(k = 1, 2)$. Hence, for the substitution of (2.5) into (2.3), the sum outside the integral sign is equal to

$$\sum_{k=1}^{2} \Big(n_1 q_k + n_2 \frac{p_k}{\mu_k}\Big)\alpha_k, \tag{2.6}$$

and in Eq. (2.4),

$$\sum_{k=1}^{2} \Big(n_2 q_k - n_1 \frac{p_k}{\mu_k}\Big)\alpha_2. \tag{2.7}$$

Let us equate the sum (2.6) to the real functions $f_1(s)$, and the sum (2.7) to the real function $f_2(s)$. We obtain the system of equations

$$\begin{aligned} \sum_{k=1}^{2} \Big(n_2 q_k - n_1 \frac{p_k}{\mu_k}\Big)\alpha_k &= f_1(s), \\ \sum_{k=1}^{2} \Big(n_2 q_k - n_1 \frac{p_k}{\mu_k}\Big)\alpha_k &= f_2(s). \end{aligned} \tag{2.8}$$

It is easy to show that the determinant at the unknowns in (2.8) does not depend on the normal vector and is equal to

$$\delta = \frac{p_2 q_1}{\mu_2} - \frac{p_1 q_2}{\mu_1}.$$

This determinant is proportional to the difference $\mu_2 - \mu_1$ and is different from zero because of the positive definiteness of the flexural rigidity matrix. Hence,

$$\delta\alpha_2 = -(p_1/\mu_1)\,(n_1 f_1(s) + n_2 f_2(s)) + q_1(n_2 f_1(s) - n_1 f_2(s)),$$

$$\delta\alpha_1 = (p_2/\mu_2)(n_1 f_1(s) + n_2 f_2(s)) - q_2(n_2 f_1(s) - n_1 f_2(s)).$$

Thus, from the previous construction it follows that $\varphi_k''(z_k)$ $(k = 1, 2)$ are single-valued functions of the coordinates. In this case, $\varphi_k'(z_k)$ $(k = 1, 2)$ are represented as

$$\begin{aligned} \varphi_1'(z_1) &= -\frac{1}{\pi i \delta}\int\limits_{\partial Q} \Big(g_1(s)\,\frac{p_2}{\mu_2} - q_2 g_2(s)\Big)\ln(z_1 - t_1)\, ds + D_1, \\ \varphi_2'(z_2) &= -\frac{1}{\pi i \delta}\int\limits_{\partial Q} \Big(-g_1(s)\,\frac{p_1}{\mu_1} + q_1 g_2(s)\Big)\ln(z_2 - t_2)\, ds + D_2. \end{aligned} \tag{2.9}$$

Here $g_1(s) = n_1 f_1(s) + n_2 f_2(s)$ and $g_2(s) = n_2 f_1(s) - n_1 f_2(s)$.

In (2.9), we choose a definite branch of the logarithm, for example, the principal one. The functions $\varphi_k'(z_k)$ $(k = 1, 2)$, generally speaking, are multivalued since in circulation around the boundary, the logarithm acquires the increment $2\pi i$. Let

$$k_i = \int_{\partial Q} g_i(s)\, ds, \qquad i = 1, 2.$$

For the functions $\varphi_k'(z_k)$ $(k = 1, 2)$ to be single-valued, it is necessary and sufficient that their increments vanish in circulation around the boundary. Obviously, this leads to the following system of equations:

$$\frac{p_2}{\mu_2}\, k_1 - q_2 k_2 = 0, \qquad -\frac{p_1}{\mu_1}\, k_1 + q_1 k_2 = 0.$$

The determinant at the unknowns k_1 and k_2 coincides with δ and is assumed to be nonzero. Hence, $k_1 = k_2 = 0$ and, consequently, the densities f_1 and f_2 should be such that

$$\int_{\partial Q} (n_1 f_1(s) + n_2 f_2(s))\, ds = 0, \qquad \int_{\partial Q} (n_2 f_1(s) - n_1 f_2(s))\, ds = 0.$$

We note that these conditions coincide in form with the last two resolvability conditions of the boundary-value problem (1.8), (1.9). This coincidence is not accidental.

3. Let us construct a system of regular integral equations for the problem considered. We have

$$-(G(x_1, x_2) + M_t(x_1, x_2)) = \operatorname{Re}\Big[\frac{q_1 t_1'(s_0)}{\pi i \delta} \int_{\partial Q} \Big(g_1(s)\, \frac{p_2}{\mu_2} - q_2 g_2(s)\Big) \frac{ds}{t_1 - z_1}\Big]$$

$$+ \operatorname{Re}\Big[\frac{q_2 t_2'(s_0)}{\pi i \delta} \int_{\partial Q} \Big(-g_1(s)\, \frac{p_1}{\mu_1} + q_1 g_2(s)\Big) \frac{ds}{t_2 - z_2}\Big]. \tag{3.1}$$

We rearrange the terms in (3.1):

$$\operatorname{Re} \frac{t_1'(s_0)}{\pi i} \int_{\partial Q} g_1(s)\, \frac{ds}{t_1 - z_1}.$$

Then,

$$-(G(x_1, x_2) + M_t(x_1, x_2)) = \operatorname{Re} \int_{\partial Q} g_1(s)\, \frac{t_1'(s_0)\, ds}{t_1 - z_1} + \operatorname{Re} \frac{q_2}{\pi i \delta} \int_{\partial Q} \Big(\frac{p_1}{\mu_1}\, g_1 - q_1 g_2\Big) \Big[\frac{t_1'(s_0)}{t_1 - z_1} - \frac{t_2'(s_0)}{t_2 - z_2}\Big]\, ds.$$

Similarly, we obtain

$$-M_n = \operatorname{Re} \int_{\partial Q} g_1(s)\, \frac{t_1'(s_0)\, ds}{t_1 - z_1} + \operatorname{Re} \frac{p_1}{\pi i \delta} \int_{\partial Q} \Big(\frac{p_2}{\mu_2}\, g_1 - q_2 g_2\Big) \Big(\frac{t_1'(s_0)}{t_1 - z_1} - \frac{t_2'(s_0)}{t_2 - z_2}\Big)\, ds.$$

Next, we assume

$$K(s, s_0) = \frac{t_1'(s_0)}{t_1 - t_{10}} - \frac{t_2'(s_0)}{t_2 - t_{20}},$$

$$K_{11} g_1 = \operatorname{Re} \frac{q_2 p_1}{\pi i \delta \mu_1} \int_{\partial Q} g_1(s) K(s, s_0)\, ds, \qquad K_{12} g_2 = -\operatorname{Re} \frac{q_2 q_1}{\pi i \delta} \int_{\partial Q} g_2(s) K(s, s_0)\, ds,$$

$$K_{21} g_1 = \operatorname{Re} \frac{p_2 p_1}{\pi i \delta \mu_2} \int_{\partial Q} g_1(s) K(s, s_0)\, ds, \qquad K_{22} g_2 = -\operatorname{Re} \frac{p_2 q_1}{\pi i \delta} \int_{\partial Q} g_2(s) K(s, s_0)\, ds.$$

Here $t_{k0} = x_1'(s_0) + \mu_k x_2'(s_0)$, $k = 1, 2$.

As a result, we have the following system of the integral equations on the boundary:

$$\begin{aligned} g_1(s_0) + K_{11}g_1 + K_{12}g_2 &= n_1b_1(s_0) + n_2b_2(s_0), \\ g_2(s_0) + K_{21}g_1 + K_{22}g_2 &= n_2b_1(s_0) - n_1b_2(s_0). \end{aligned} \tag{3.2}$$

System (3.2) is the required one. It is easy to show that for its resolvability, the boundary data should satisfy the conditions

$$\int\limits_{\partial Q} (n_2g_1(s) - n_1g_2(s))\,ds = 0, \qquad \int\limits_{\partial Q} (n_1g_1(s) + n_2g_2(s))\,ds = 0. \tag{3.3}$$

Indeed, we multiply (3.2) by ds_0 and integrate over s_0 taking into account that

$$\int\limits_{\partial Q} \frac{t_k'(s_0)\,ds_0}{t_k(s) - t_k(s_0)} = -\pi i, \qquad k = 1, 2.$$

Then, the left side of Eqs. (3.2) vanish and conditions (3.3) remain.

The system of equations coupled (after Fredholm) to system (3.2) is written as

$$\begin{aligned} m_1(s_0) - \operatorname{Re}\frac{1}{\pi i}\int\limits_{\partial Q} m_1(s)\,\frac{dt_1}{t_1 - t_{10}} + \frac{p_1}{\mu_1\pi i\delta}\int\limits_{\partial Q}\Big(q_2m_1 + \frac{p_2}{\mu_2}\,m_2\Big)\Big(\frac{dt_2}{t_2 - t_{20}} - \frac{dt_1}{t_1 - t_{10}}\Big) &= r_1(s_0), \\ m_2(s_0) - \operatorname{Re}\frac{1}{\pi i}\int\limits_{\partial Q} m_2(s)\,\frac{dt_2}{t_2 - t_{20}} - \frac{q_2}{\pi i\delta}\int\limits_{\partial Q}\Big(q_1m_1 + \frac{p_1}{\mu_1}\,m_2\Big)\Big(\frac{dt_2}{t_2 - t_{20}} - \frac{dt_1}{t_1 - t_{10}}\Big) &= r_2(s_0). \end{aligned} \tag{3.4}$$

Equations (3.4) are related to the functions

$$v_1 = \operatorname{Re}\frac{p_2}{\mu_2\pi i\delta}\int\limits_{\partial Q}\Big(q_1m_1 + \frac{p_1}{\mu_1}\,m_2\Big)\frac{dt_1}{t_1 - z_1} - \operatorname{Re}\frac{p_1}{\mu_1\pi i\delta}\int\limits_{\partial Q}\Big(q_2m_1 + \frac{p_2}{\mu_2}\,m_2\Big)\frac{dt_2}{t_2 - z_2},$$

$$v_2 = -\operatorname{Re}\frac{q_2}{\pi i\delta}\int\limits_{\partial Q}\Big(q_1m_1 + \frac{p_1}{\mu_1}\,m_2\Big)\frac{dt_1}{t_1 - z_1} + \operatorname{Re}\frac{q_1}{\pi i\delta}\int\limits_{\partial Q}\Big(q_2m_1 + \frac{p_2}{\mu_2}\,m_2\Big)\frac{dt_2}{t_2 - z_2},$$

where $m_k(s)$ $(k = 1, 2)$ are the real densities. The functions v_k $(k = 1, 2)$ can be treated as a solution of the Dirichlet problem for the elliptic equations in the external domain Q_e. In this case, system (3.4) has the eigenfunctions $m_1(s) = 1$ and $m_2(s) = 1$. It does not have other eigenfunctions. We recall that a system of equations coupled to a Fredholm system of equations is also a Fredholm system.

Let us examine whether the constructed system of integral equations is adequate to the initial boundary-value problem. It was shown above that for the system of integral equations, two resolvability conditions hold and for the initial boundary-value problem, three resolvability conditions should be satisfied. Hence, the constructed system of equations is adequate only to the modified rather than the initial problem. The reasons for this are clear. Indeed, in the initial boundary-value problem, the combination $\partial\,(M + M_t)/\partial s$ is specified on the boundary, whereas (2.4) contains only the combination $M + M_t$. Adequacy to the initial boundary-value problem will take place if $f_1(s)$ is taken in the form

$$f_1(s) = \int\limits_0^s \tilde{f}_1(t)\,dt$$

subject to the condition

$$\int\limits_0^L \tilde{f}_1(t)\,dt = 0.$$

As a result, all three resolvability conditions are satisfied. What can be said of the smoothness of the solution of the problem? Clearly, the second derivatives of the solution are continuous up to the boundary of the domain. The generalized shear force is also continuous. However, for the existence of the third derivatives of the solution on the boundary, it is necessary to require that

$$x_k(s) \in C^{2,\lambda}(0,L), \qquad f_k(s) \in C^{1,\lambda}(0,L), \qquad k=1,2.$$

Similarly to [1], one can prove that system (3.2) is indeed a Fredholm one, i.e., the kernels of the integral operators appearing in this system have at most a weak singularity if the boundary satisfies Lyapunov's condition.

4. We construct one more system of integral equations. For the examined boundary-value problem, a uniquely solvable system of integral equations can be written assuming that the boundary data satisfy the equilibrium conditions. Indeed, integrating boundary-value conditions (2.2) and (2.3) once more along the arc length, we obtain

$$\operatorname{Re}\sum_{k=1}^{2} q_k\varphi_k'(z_k)\Big|_{\partial Q} = \int_0^{s_0}\Big(-n_1\Big(\int_0^{s} h_1(t)\,dt + C_1\Big) + n_2h_2(s)\Big)\,ds + D_1; \tag{4.1}$$

$$\operatorname{Re}\sum_{k=1}^{2} \frac{p_k}{\mu_k}\,\varphi_k'(z_k)\Big|_{\partial Q} = -\int_0^{s_0}\Big(n_2\Big(\int_0^{s} h_1(t)\,dt + C_1\Big) + n_1h_2(s)\Big)\,ds + D_2. \tag{4.2}$$

We set

$$g_1(s_0) = \int_0^{s_0}\Big(-n_1\Big(\int_0^{s} h_1(t)\,dt + C_1\Big) + n_2h_2(s)\Big)\,ds,$$

$$g_2(s_0) = \int_0^{s_0}\Big(n_2\Big(\int_0^{s} h_1(t)\,dt + C_1\Big) + n_1h_2(s)\Big)\,ds.$$

Since the functions $\varphi_k'(z_k)$ $(k=1,2)$ should be one-valued functions of the coordinates, equalities $g_k(L)=0$ $(k=1,2)$ should be satisfied, which are equivalent to the equilibrium conditions. A system of integral equations for the boundary-value problem (4.1), (4.2) can be constructed by analogy with the approach described above. We set

$$\varphi_k'(z_k) = \frac{1}{\pi i}\int_{\partial Q}\frac{\beta_k\,dt_k}{t_k - z_k}, \qquad k=1,2.$$

We determine the densities β_k $(k=1,2)$ from the system of equations

$$q_1\beta_1 + q_2\beta_2 = f_{01}, \qquad p_1\beta_1/\mu_1 + p_2\mu_2\beta_2 = f_{02}, \tag{4.3}$$

where $f_{0k}(s)$ $(k=1,2)$ are real functions. Solving system (4.3), we obtain

$$\varphi_1'(z_1) = \frac{1}{\pi i\delta}\int_{\partial Q}\Big(\frac{p_2}{\mu_2}\,f_{01} - q_2f_{02}\Big)\frac{dt_1}{t_1 - z_1}, \qquad \varphi_2'(z_2) = \frac{1}{\pi i\delta}\int_{\partial Q}\Big(-\frac{p_1}{\mu_1}\,f_{01} + q_1f_{02}\Big)\frac{dt_2}{t_2 - z_2}.$$

From this, it follows that

$$\operatorname{Re}\sum_{k=1}^{2} q_k\varphi_k'(z_k) = \frac{1}{\pi i}\int_{\partial Q} f_{01}\,\frac{dt_1}{t_1 - z_1} + \frac{1}{\pi i\delta}\int_{\partial Q}\Big(\frac{p_1q_2}{\mu_1}\,f_{01} - q_1q_2f_{02}\Big)\Big(\frac{dt_1}{t_1 - z_1} - \frac{dt_2}{t_2 - z_2}\Big),$$

$$\operatorname{Re}\sum_{k=1}^{2} \frac{p_k}{\mu_k}\,\varphi_k'(z_k) = \frac{1}{\pi i}\int_{\partial Q} f_{02}\,\frac{dt_2}{t_2 - z_2} + \frac{1}{\pi i\delta}\int_{\partial Q}\Big(\frac{p_1p_2}{\mu_1\mu_2}\,f_{01} - \frac{p_1q_2}{\mu_1}\,f_{02}\Big)\Big(\frac{dt_1}{t_1 - z_1} - \frac{dt_2}{t_2 - z_2}\Big).$$

As a result, we have the following system of regular integral equations:

$$f_{01}(s_0) + \frac{1}{\pi i}\int_{\partial Q} f_{01}\,\frac{dt_1}{t_1 - t_{10}} + \frac{1}{\pi i\delta}\int_{\partial Q}\Big(\frac{p_1q_2}{\mu_1}\,f_{01} - q_1q_2f_{02}\Big)\Big(\frac{dt_1}{t_1 - t_{10}} - \frac{dt_2}{t_2 - t_{20}}\Big) = g_1(s_0),$$

$$f_{02}(s_0) + \frac{1}{\pi i} \int\limits_{\partial Q} f_{02} \frac{dt_2}{t_2 - t_{20}} + \frac{1}{\pi i \delta} \int\limits_{\partial Q} \Big(\frac{p_1 p_2}{\mu_1 \mu_2} f_{01} - \frac{p_1 q_2}{\mu_1} f_{02} \Big) \Big(\frac{dt_1}{t_1 - t_{10}} - \frac{dt_2}{t_2 - t_{20}} \Big) = g_2(s_0).$$

In contrast to the system of equations constructed above, this system is uniquely solvable.

5. We study the limiting transition to an isotropic material. It is easy to show that the constant δ is proportional to the difference $\mu_1 - \mu_2$. Let

$$\delta = \frac{\mu_1 - \mu_2}{\mu_1 \mu_2} \chi.$$

In this case,

$$\chi = 2D_{11}D_{12} + (D_{11}D_{22} - D_{12}^2)\mu_1\mu_2 + 2D_{26}D_{11}(\mu_1 + \mu_2)$$

$$+ 2D_{16}D_{26}\mu_1\mu_2 + 2D_{16}D_{22}\mu_1\mu_2(\mu_1 + \mu_2).$$

The positive definiteness of the elastic constant matrix implies that the constant χ is not equal to zero. Thus, for an orthotropic material, ($D_{16} = D_{26} = 0$), we have

$$\chi = 4D_{11}D_{66} + (D_{11}D_{22} - D_{12}^2)\sqrt{D_{11}/D_{22}}.$$

In particular, for an isotropic material, we have

$$\chi = (1-\nu)(3+\nu)/(1-\nu^2)^2$$

(ν is Poisson's factor). Passing to the limit, we obtain

$$\frac{p_1 p_2}{\chi} = \frac{1-\nu}{3+\nu}, \qquad \frac{p_1 q_2 \mu_2}{\chi} = -\frac{i(1-\nu)}{3+\nu}.$$

In this case, the difference of integrals

$$\frac{t_1'(s_0)}{\pi i(\mu_1 - \mu_2)} \int\limits_{\partial Q} \frac{f(s)\,ds}{t_1 - z_1} - \frac{t_2'(s_0)}{\pi i(\mu_1 - \mu_2)} \int\limits_{\partial Q} \frac{f(s)\,ds}{t_2 - z_2}$$

becomes the integral

$$\frac{1}{\pi i} \int\limits_{\partial Q} f(s) \frac{x_2'(s_0)(x_1(s) - x_1) - x_1'(s_0)(x_2(s) - x_2)}{(t - z)^2}\, ds.$$

Setting

$$m(s, s_0, z) = \frac{x_2'(s_0)(x_1(s) - x_1) - x_1'(s_0)(x_2(s) - x_2)}{(t - z)^2}.$$

we obtain the following system of equations for an isotropic material:

$$\alpha_1(s_0) + \mathrm{Re}\, \frac{t_1'(s_0)}{\pi i} \int\limits_{\partial Q} \alpha_1(s) \frac{ds}{t - t_0} + \frac{1-\nu}{3+\nu}\, \mathrm{Re}\, \frac{1}{\pi} \int\limits_{\partial Q} (\alpha_1(s) + i\alpha_2(s)) m(s, s_0, t_0)\, ds = n_1 g_1(s_0) + n_2 g_2(s_0),$$

$$\alpha_2(s_0) + \mathrm{Re}\, \frac{t_2'(s_0)}{\pi i} \int\limits_{\partial Q} \alpha_2(s) \frac{ds}{t - t_0} - \frac{1-\nu}{3+\nu}\, \mathrm{Re}\, \frac{1}{\pi i} \int\limits_{\partial Q} (\alpha_1(s) + i\alpha_2(s)) m(s, s_0, t_0)\, ds = n_2 g_1(s_0) - n_1 g_2(s_0).$$

6. The examined boundary-value problem is similar to the second boundary-value problem in elasticity theory (the force vector is specified on the boundary). As is known, in the second boundary-value elasticity problem, the displacement vector is determined only with accuracy up to the rigid displacement vector and its solution should satisfy the equilibrium conditions (the resultant vector and the resultant force moment are equal to zero). The equilibrium conditions in this problem do not coincide with the equilibrium conditions in elasticity theory. The fact is that the solution of the homogeneous boundary-value problem is a linear function. Indeed, the homogeneous equation of plate bending can be written in symmetric form as a system of three equations:

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} = N_{11}, \quad \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} = N_{22}, \quad \frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} = 0. \tag{6.1}$$

Obviously, the first two equations of system (6.1) are similar to the equations of the two-dimensional systems of elasticity equations. Multiplying the first equation by v_1 and the second equation by v_2, combining them, and integrating by parts, we obtain the equality

$$-\int_Q M_{ij} \frac{\partial v_i}{\partial x_j}\, dx + \int_{\partial Q} M_{ij} v_i n_j \, ds = \int_Q (N_{11} v_1 + N_{22} v_2)\, dx. \tag{6.2}$$

In (6.2), the summation is performed over repeated indices from 1 to 2. If N_{ii} $(i = 1, 2)$ were equal to zero, the standard system of elasticity equations should hold and for the moment vector specified on the boundary $M_{ij} n_j$ $(i, j = 1, 2)$, the equilibrium conditions had the form of the equality to zero of the resultant moment and the resultant force vector. However, this is prevented by the presence of the right side in (6.2). Setting $v_1 = \partial v/\partial x_1$ and $v_2 = \partial v/\partial x_2$ and integrating the right side in (6.2) by parts, we obtain

$$\int_Q \Big(N_{11} \frac{\partial v}{\partial x_1} + N_{22} \frac{\partial v}{\partial x_2} \Big)\, dx_1\, dx_2 = \int_{\partial Q} (N_{11} n_1 + N_{22} n_2) v \, ds.$$

The integral over the domain Q on the right side vanishes by virtue of the third equation of system (6.1). These calculations show that the equilibrium conditions in the examined problem do not coincide with the classical equilibrium conditions. However, the fact that the given problem does not belong to the class of uniquely solvable problems and its solution should satisfy the equilibrium conditions is not noted and used in [5, 6]. Even for an isotropic material there are considerable differences in the form of the boundary conditions of this problem in [5] and [6].

REFERENCES

1. Yu. A. Bogan, "On the potential method for the elliptic equation from the anisotropic theory of elasticity," *Sib. Zh. Indust. Mat.*, **3**, No. 2, 29–34 (2000).
2. K. Rektorys, *Variational Methods in Mathematics, Science, and Engineering*, Reidel, Dordrecht (1983).
3. S. G. Lekhnitskii, "Some issues related to the theory of bending of thin plates," *Prikl. Mat. Mekh.*, **2**, No. 2, 181–210 (1938).
4. Yu. A. Bogan, "Integral Fredholm equations in the two-dimensional anisotropic theory of elasticity," *Sib. Zh. Vychisl. Mat.*, **4**, No. 1, 21–30 (2002).
5. L. T. Berezhnitskii, M. V. Delyavskii, and V. I. Panasyuk, *Bending of Thin Plates with Crack-Type Defects* [in Russian], Naukova Dumka, Kiev (1979).
6. I. A. Prusov, *Conjugation Method in Plate Theory* [in Russian], Izd. Belorus. Univ, Minsk (1975).